A family of $K3$ surfaces and $\zeta(3)$

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1. Introduction

Let

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right\} \qquad (n=1,2,3,\ldots).$$

It was R. Apéry's surprising discovery, that $\frac{b_n}{a_n}$ tends to $\zeta(3)=\sum_1^\infty m^{-3}$ as $n \to \infty$ and the convergence is fast enough to prove irrationality of $\zeta(3)$, see [P]. Another remarkable discovery of Apéry is, that a_n and b_n satisfy the recurrence relation

$$(1) \qquad (n+1)^3 u_{n+1} = (34n^3+51n^2+27n+5)u_n - n^3 u_{n-1}$$

(see [P]). The first few terms of $\{a_n\}$ and $\{b_n\}$ are given by

$$\{a_n\} = 1, 5, 73, 1445, 33001, 819005, \ldots,$$

$$\{b_n\} = 0, 6, \frac{351}{4}, \ldots .$$

Consider the generating functions

$$\mathscr{A}(t) = \sum_0^\infty a_n t^n, \quad \mathscr{B}(t) = \sum_0^\infty b_n t^n \quad \text{and} \quad \mathscr{R}(t) = \mathscr{A}(t)\,\zeta(3) - \mathscr{B}(t).$$

As a consequence of (1) these functions satisfy the differential equations

$$(2) \qquad L\mathscr{A}(t)=0, \quad L\mathscr{B}(t)=5, \quad L\mathscr{R}(t)=-5$$

where L is the differential operator given by

$$(3) \qquad L=(t^4-34t^3+t^2)\left(\frac{d}{dt}\right)^3+(6t^3-153t^2+3t)\left(\frac{d}{dt}\right)^2+(7t^2-112t+1)\frac{d}{dt}+(t-5).$$

There has been a strong suspicion that the equation $Ly=0$ and a similar equation corresponding to Apéry's irrationality proof for $\zeta(2)$, "must come from algebraic geometry". For the equation corresponding to $\zeta(2)$ the first author has shown that it is indeed the Picard-Fuchs equation of a modular family of elliptic curves [Be 2]. In this paper we prove that $Ly=0$ is in fact the Picard-Fuchs equation of the family of algebraic surfaces given by

$$1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)=0 \tag{4}$$

where t is the parameter of the family. For the definition of Picard-Fuchs equation, Gauss-Manin connexion, etc. see [Ka 1, 2]. The central result of this paper is that, for general t, a surface given by (4) is birationally equivalent to a $K3$ surface with Picard number 19.

Before stating the Main Theorem and its consequences in section 3 we give a brief overview of the terminology concerning $K3$ surfaces in section 2. In section 4 we then prove the Main Theorem.

We close this introduction with a word of apology to the reader who is not well versed in algebraic geometry. In order not to lose the number theoretical reader too soon, we have tried to give the proofs in an extensive manner with references for further reading. Furthermore, we hope that the Main Theorem and its Corollaries contain sufficiently little geometry in order to pass their content also to the non-geometer.

2. Some properties of $K3$ surfaces

A $K3$ surface X can be defined as a compact, simply connected non-singular complex-analytic surface having a trivial canonical class. Examples of such surfaces are given by non singular surfaces of degree 4 in $\mathbb{P}^3$, or by twofold coverings of $\mathbb{P}^2$ branched along a nonsingular curve of degree 6.

Since X is simply connected the cohomology groups $H^i(X, \mathbb{C})$ are trivial for $i=1,3$ and have rank 1 if $i=0,4$. The group $H^2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. Endowed with the bilinear symmetric form $\langle\, ,\rangle$ coming from the cup product, it becomes a unimodular lattice of signature $(3,19)$. Up to a scalar multiple, there is a unique holomorphic 2-form ω_X on X. The cohomology class $[\omega_X]$ of X spans the subspace $H^{2,0}(X, \mathbb{C})$ of $H^2(X, \mathbb{C})$ and satisfies the well-known relations $\langle[\omega_X],[\omega_X]\rangle=0$ and $\langle[\omega_X],[\bar{\omega}_X]\rangle>0$.

The collection of all $K3$ surfaces is parametrised by a connected 20-dimensional manifold M. However, the algebraic $K3$ surfaces form a countable union of 19-dimensional irreducible subspaces. Furthermore, those algebraic $K3$ surfaces having k or more divisors independent in homology form a dense countable union of subvarieties of dimension $20-k$ inside M.

Let X be an algebraic $K3$ surface. The subgroup of $H_2(X, \mathbb{Z})$ generated by the divisors of X is the Néron-Severi group of X and its rank is called the Picard number of X, denoted by $\rho(X)$. We have that $\rho(X) \leq h^{1,1} = \mathrm{rank}_{\mathbb{C}}\, H^{1,1}(X, \mathbb{C}) = 20$. $K3$ surfaces with $\rho(X)=20$ form a discrete countable family. They were classified in [Sh 1]. In this paper we shall be interested in a (non constant) family of $K3$ surfaces with Picard number 19 which arises naturally in connection with Apéry's irrationality proof of $\zeta(3)$. From the foregoing it follows that its parameterset maps to one of the countably many irreducible curves in M corresponding to $K3$'s with Picard number $\geqq 19$.

For more information on $K3$ surfaces one may consult for example [GH], p. 590, [BPV], Ch. VIII or [LP].

3. The results

Let a_n be the numbers defined in the introduction. Let $P_n(x)$ be the polynomial of degree n given by

$$P_n(x)=\frac{1}{n!}\left(\frac{d}{dx}\right)^n x^n(1-x)^n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-x)^k.$$

Notice that

$$a_n=\sum\binom{n}{k}^2\binom{n+k}{k}^2=\frac{1}{2\pi i}\int_{|x|=1}\frac{1}{x}P_n(x)\,P_n\left(\frac{1}{x}\right)dx.$$

By Cauchy's residue theorem the latter integral equals

$$-\left(\frac{1}{2\pi i}\right)^2\int_{\substack{|x|=1\\|y|=2}}\frac{P_n(x)\,P_n(y)}{1-xy}\,dx\,dy$$

and this equals

$$\left(\frac{1}{2\pi i}\right)^3\int_{\substack{|x|=1\\|y|=2\\|z|=2}}\frac{P_n(x)\,P_n(y)}{1-(1-xy)z}\,dx\,dy\,dz.$$

Following the same arguments as in [Be 1, p. 271] we see that this integral equals

$$a_n=\left(\frac{1}{2\pi i}\right)^3\int_S\frac{X^n(1-X)^n\,Y^n(1-Y)^n\,Z^n(1-Z)^n}{(1-(1-XY)Z)^{n+1}}\,dX\,dY\,dZ$$

where the integration is over a suitable 3-dimensional closed integration area S. Hence

$$\mathscr{A}(t)=\left(\frac{1}{2\pi i}\right)^3\int_S\frac{dX\,dY\,dZ}{1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)}\,.$$

Denote the projective algebraic surface with affine equation

$$1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)=0$$

by S_t, and denote the differential form in the expression for $\mathscr{A}(t)$ by Ω_t. The surface S_t is exactly the polar locus of Ω_t. A straightforward computation shows that the three lines at infinity given by $X=0$, $Y=0$, $Z=0$ respectively, consist of singular points of S_t. If $t\notin\{0,\infty,(\sqrt{2}\pm 1)^4\}$ there are no other singular points on S_t. If $t=0$ or ∞ then S_t becomes reducible. If $t=(\sqrt{2}\pm 1)^4$, the surface S_t acquires a finite double point.

In retrospect, we have proved the following,

Proposition. *Let S_t, Ω_t be as defined above. Then there exists a closed real three-dimensional area S in $\mathbb{C}^3 \setminus S_t$ such that $\mathscr{A}(t)=(2\pi i)^{-3}\int_S \Omega_t$. Moreover, by* (2), *$\mathscr{A}(t)$ satisfies the differential equation $L\mathscr{A}=0$.*

We now show that the integral $\int_S \Omega_t$ where S is a closed three dimensional area in $\mathbb{C}^3 \setminus S_t$, is in fact a period of a two form on S_t. To this end we use the procedure described in Griffiths [Gr], p. 470. In this procedure we rely on the fact that as a consequence of Stokes' theorem, $\int_S \Omega_t$ depends only on the homology class of S in $H_3(\mathbb{C}^3 \setminus S_t, \mathbb{Z})$. We now shrink S to a small tube around a 2-cycle on S_t as follows. Write $S=\partial R$, where R is a 4-chain in $\mathbb{C}^3$. We may assume that R is in general position with respect to S_t and thus R meets S_t transversely in a 2-cycle γ lying on S_t (since $\partial(R \cdot S_t)=\partial R \cdot S_t \pm R \cdot \partial S_t=0$). Let $T_\varepsilon(\gamma)$ be the set of points z of R with distance $(z,\gamma)<\varepsilon$. Because of transversality, for sufficiently small ε a point z in $T_\varepsilon(\gamma)$ is given uniquely by a pair (v, w) where v lies on γ and w belongs to the normal ε-disc to S_t at v. Thus $T_\varepsilon(\gamma)$ is a solid tube of discs surrounding γ. The boundary $\partial T_\varepsilon(\gamma)=T_\varepsilon(\gamma)$ is a family of disjoints circles lying in $\mathbb{C}^3 \setminus S_t$ and parametrised by γ. Clearly

$$\partial(R-T_\varepsilon(\gamma))=S-\tau_\varepsilon(\gamma)$$

so that S and $\tau_\varepsilon(\gamma)$ are in the same homology class. We call $\tau_\varepsilon(\gamma)$ the ε-tube lying over γ, and $\tau(\gamma)$ will be any $\tau_\varepsilon(\gamma)$ for ε sufficiently small. Thus we have, to every $S \in H_3(\mathbb{C}^3 \setminus S_t, \mathbb{Z})$ corresponds a finite 2-cycle $\gamma \in H_2(\mathbb{C}^3 \cap S_t, \mathbb{Z})$ such that

$$\int_S \Omega_t = \int_{\tau(\gamma)} \Omega_t .$$

By the Poincaré residue theorem we now have

$$\begin{aligned}\int_{\tau(\gamma)} \Omega_t &= 2\pi i \int_\gamma \frac{1}{2} \frac{dX \wedge dZ}{\frac{\partial}{\partial Y}\{1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)\}} \\ &= \pi i \int_\gamma \frac{dX \wedge dZ}{XZ(1-t(1-X)(1-Z)(1-2Y))} .\end{aligned}$$

Define the (rational) 2-form ω_t on S_t by

$$\omega_t = \left.\frac{dX \wedge dZ}{XZ(1-t(1-X)(1-Z)(1-2Y))}\right|_{S_t} . \tag{5}$$

We now state our

Main theorem. *Let $t \notin \{0, 1, (\sqrt{2} \pm 1)^4, \infty\}$. Then,*

i) *The algebraic surface S_t given by*

$$1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)=0$$

is birationally equivalent to a K3 surface X_t.

ii) *The form ω_t given by* (5) *is, apart from a scalar factor, the unique holomorphic 2-form on X_t.*

iii) *The Picard number satisfies*: $\rho(X_t) \geqq 19$ *and equality holds for all but countably many t.*

Before proving this theorem in section 4 we state and prove two corollaries.

Corollary 1. *The periods $\int_\gamma \omega_t$, where $\gamma \in H_2(S_t, \mathbb{Z})$, span the space of solutions of the differential equation $Ly=0$, where L is the linear differential operator defined in* (3).

Corollary 2. *Three solutions of $Ly=0$ around $t=0$ are given by*

$$y_0 = \mathscr{A}(t),$$
$$y_1 = \mathscr{A}(t)\log t - 3 - 3t - 9t^2 + 103t^3 + 5822t^4 + \cdots,$$
$$y_2 = -\frac{1}{2}\mathscr{A}(t)\log^2 t + y_1 \log t - 36t - 558t^2 - 11154t^3 - 255225t^4 - \cdots$$

where $\mathscr{A}(t)$ is the generating function defined in the introduction. Moreover, they satisfy the relation

$$2y_0y_2 + 9y_0^2 = y_1^2.$$

Remark. Although the expansions of y_1, y_2 in powers of t seem to suggest it, not all coefficients are in $\mathbb{Z}$.

For the proof of Corollary 1 we need a

Lemma. *Let L be the differential operator defined in* (3). *Then*

i) *L is irreducible, that is, L cannot be written as the product of two differential operators of order $\geqq 1$ with rational coefficients.*

ii) *the $\mathbb{C}$-linear space spanned by all branches obtained by analytic continuation of a nontrivial solution of $Ly=0$ has dimension* 3.

Proof. i) Notice that $Ly=0$ is a Fuchsian differential equation with local exponents $(0,0,0)$ at $t=0$, $(0,\frac{1}{2},1)$ at $t=(\sqrt{2}\pm 1)^4$ and $(1,1,1)$ at $t=\infty$. For the definition of "Fuchsian", local exponent, etc. see Ince [In], Ch. XV. Suppose L can be written as $L=L_1 \cdot L_2$ where L_1, L_2 are linear differential operators of order $\geqq 1$ with coefficients in $\mathbb{C}(t)$. Let $p \in \mathbb{P}_1(\mathbb{C})$. It is clear that the local exponents of $L_2 y=0$ at p form a subset of those for $Ly=0$. Morever, the non-apparent singularities of $L_2 y=0$ form a subset of $\{0, (\sqrt{2}\pm 1)^4, \infty\}$. We now recall Fuchs' formula [In], Ch. XV

$$\sum_{p \in P(C)} \left\{\sigma_p - \frac{1}{2}k(k-1)\right\} = -k(k-1) \tag{6}$$

where k is the order of a Fuchsian differential equation and σ_p is the sum of its local exponents at p. Only for the singularities of the differential equation the summand is non-zero. We apply this formula to $L_2 y=0$. First let $k=$ order of L_2 be one. Then (6) implies $\sum_p \sigma_p = 0$. However $\sigma_\infty = 1$ and since $\sigma_p \geqq 0$ for all other p, we arrive at a contradiction. Now let k = order of L_2 be two. Then (6) implies $\sum_p (\sigma_p - 1) = -2$. We now have $\sigma_\infty = 2$, $\sigma_p \geqq \frac{1}{2}$ for $p = (\sqrt{2} \pm 1)^4$, $\sigma_0 = 0$ and $\sigma_p - 1 \geqq 0$ for all other p. Hence $1 + 2\left(\frac{1}{2} - 1\right) + (-1) \leqq -2$, which is clearly a contradiction. We thus conclude that L is irreducible.

ii) Let w be a non-trivial solution of $Ly = 0$ such that its branches obtained by analytic continuation span a $\mathbb{C}$-linear space V of dimension < 3. Then w must satisfy a linear differential equation $\tilde{L}y = 0$ of order < 3 with coefficients in $\mathbb{C}(t)$. Choose the order of $\tilde{L}$ minimal. Then, by application of a euclidean algorithm to L and $\tilde{L}$ where we use the order as a norm, we find that we must have $L = L' \cdot \tilde{L}$ for a certain $L' \in \mathbb{C}(t)\left[\frac{d}{dt}\right]$. According to part i) of this Lemma this is impossible, hence V must have dimension 3, as asserted.

Proof of Corollary 1. X_t is a nonsingular model of S_t and we denote by ω_t the unique holomorphic 2-form on X_t. By Proposition 1 we know that there exists a $\gamma_0 \in H_2(X_t, \mathbb{Z})$ such that $\int_{\gamma_0} \omega_t$ is not identically zero and it satisfies the differential equation $Ly = 0$. The transforms of $\int_{\gamma_0} \omega_t$ obtained by analytic continuation are all of the form $\int_\gamma \omega_t$, $\gamma \in H_2(X_t, \mathbb{Z})$. Moreover, by our Lemma they span the $\mathbb{C}$-linear space of solutions of $Ly = 0$.

Since $\int_\gamma \omega_t = 0$ if γ is an algebraic 2-cycle, we conclude that there must be three 2-cycles $\gamma_1, \gamma_2, \gamma_3$, which are linearly independent modulo algebraic cycles and such that $\int_{\gamma_i} \omega_t$ $(i = 1, 2, 3)$ forms a basis of the solutions of $Ly = 0$. Since $\rho(X_t) = 19$, we can complete $\gamma_1, \gamma_2, \gamma_3$ to a $\mathbb{C}$-basis of $H_2(X_t, \mathbb{Z}) \otimes \mathbb{C}$ by adding 19 algebraic 2-cycles $\gamma_4, \ldots, \gamma_{22}$. Now let $\gamma \in H_2(X_t, \mathbb{Z})$ be arbitrary. Then $\gamma = \sum_{i=1}^{22} \lambda_i$, for certain $\lambda_i \in \mathbb{C}$ and hence $\int_\gamma \omega_t = \sum_{i=1}^{3} \lambda_i \int_{\gamma_i} \omega_t$. Thus, any period $\int_\gamma \omega_t$ satisfies $Ly = 0$.

Proof of Corollary 2. The fact that y_0, y_1, y_2 satisfy $Ly = 0$ can be checked by a simple substitution and the fact that $L\mathscr{A}(t) = 0$.

For the second statement we use the dual isomorphism between $H^2(X_t, \mathbb{Q})$ and $H_2(X_t, \mathbb{Q})$ where $\int_\gamma \omega$ is the dual pairing on $H_2(X_t, \mathbb{Q}) \times H^2(X_t, \mathbb{Q})$. On $H_2(X, \mathbb{Q})$ we have the symmetric bilinear form $(\cdot, \cdot)$ which is given by the intersection between 2-cycles. It is the dual of the quadratic form $\langle \cdot, \cdot \rangle$ on $H^2(X_t, \mathbb{Q})$. Let $\gamma_4, \gamma_5, \ldots, \gamma_{22}$ be a $\mathbb{Q}$-basis of the algebraic 2-cycles. Let $\gamma_1, \gamma_2, \gamma_3$ be a $\mathbb{Q}$-basis of the orthogonal complement of the algebraic 2-cycles. Thus we have $(\gamma_i, \gamma_j) = 0$ for $i = 1, 2, 3$, $j = 4, 5, \ldots, 22$ and the matrix $(\gamma_i, \gamma_j)_{i,j=1}$ is non-trivial and symmetrical.

By duality the relation $\langle\omega_t, \omega_t\rangle = 0$ implies

$$\sum_{i,j=1}^{22} \Big(\int_{\gamma_i} \omega_t \int_{\gamma_j} \omega_t\Big)(\gamma_i, \gamma_j) = 0. \tag{7}$$

Since $\int_{\gamma_i} \omega_t = 0$ for algebraic γ_i, (7) reduces to

$$\sum_{i,j=1}^{3} \Big(\int_{\gamma_i} \omega_t \int_{\gamma_j} \omega_t\Big)(\gamma_i, \gamma_j) = 0. \tag{8}$$

The matrix $(\gamma_i, \gamma_j)_{i,j=1}^{3}$ is symmetric, non-trivial and locally constant with respect to t. Therefore the quadratic relation (8) between the periods $\int_{\gamma_i} \omega_t$ is non-trivial and has locally constant coefficients. By the previous corollary we know that $\int_{\gamma_i} \omega_t$ is a $\mathbb{C}$-linear combination of y_0, y_1, y_2 and thus the y_i satisfy a non-trivial homogeneous quadratic relation with coefficients which are locally independent of t, hence constant. By comparison of the expressions for y_i one easily derives the desired relationship.

Remark. We can paraphrase the preceding argument as follows. Considering t as an affine coordinate on $\mathbb{P}^1$, the equation (4) defines a (singular) threefold $S \subset \mathbb{P}^3 \times \mathbb{P}^1$. The fibres of the morphism $S \to \mathbb{P}^1$ induced by the projection are exactly the S_t's.

Replacing S by a suitable non-singular model X, we get a morphism $\zeta: X \to \mathbb{P}^1$, which over $P = \mathbb{P}^1 \setminus \{0, 1, (\sqrt{2} \pm 1)^4, \infty\}$ is a family of $K3$-surfaces. There exist 19 divisors on X flat over $\mathbb{P}^1$ such that for each of the $K3$-surfaces X_t ($t \in P$) they span a rank 19 sublattice $N_t \subset H^2(X_t, \mathbb{Z})$ invariant under monodromy. So the N_t's form a subsystem of the locally constant sheaf $\{H^2(X_t, \mathbb{Z})\}$. If we lift the family $\{X_t\}$ to the universal cover $\tilde{P}$ of P we may at the same time identify the $H^2(X_u, \mathbb{Z})$'s ($u \in \tilde{P}$): $H^2(X_u, \mathbb{Z}) = H$ and $N_u := N \subset H$. The orthogonal complement $T = N^{\perp}$ of N inside H is of rank 3. The line in $H \otimes \mathbb{C}$ spanned by the cohomology class of the holomorphic 2-form ω_u remains inside $T \otimes \mathbb{C}$. So the corresponding point in $\mathbb{P}(H \otimes \mathbb{C})$ remains inside the projective 2-plane $\mathbb{P}(T \otimes \mathbb{C}) \subset \mathbb{P}(H \otimes \mathbb{C})$. Since the equation $\langle X, X\rangle = 0$ determines a non-singular quadric in $\mathbb{P}(H \otimes \mathbb{C})$, the relations $\langle[\omega_u], [\omega_u]\rangle = 0$ ($u \in \tilde{P}$) imply that, when u varies over $\tilde{P}$ the point corresponding to $[\omega_u]$ in $\mathbb{P}(H \otimes \mathbb{C})$ varies over a non-singular conic in the projective plane $\mathbb{P}(T \otimes \mathbb{C})$. It is this conic whose equation is given in the preceding corollary (with respect to a suitable basis for $T \otimes \mathbb{C}$).

4. Proof of the main theorem

The surface S_t has an affine equation

$$1 - (1 - XY)Z - tXYZ(1 - X)(1 - Y)(1 - Z) = 0.$$

Under the birational transformation of $\mathbb{P}_3$ given by

$$X = 1 + x, \quad Z = 1 + \frac{1}{y}, \quad Y = \frac{1}{2}\left(1 - \frac{y + z_1}{xt}\right)$$

we obtain a model with affine equation

$$4txy = (x + 1)(y + 1)\left(z_1^2 - (xt - y)^2\right),$$

but a more suitable model is obtained after the birational substitution

$$x=\frac{u}{1-u}, \quad y=\frac{v}{1-v}, \quad z_1=\frac{z}{(1-u)(1-v)},$$

namely

(9)
$$z^2=4tuv(1-u)^2(1-v)^2+\big(u(1-v)t-v(1-u)\big)^2.$$

The form ω_t defined by (5) transforms into

(10)
$$\omega_t=\frac{du\wedge dv}{z}.$$

Reformulating these results more geometrically, we have:

The surface S_t is birationally equivalent to a double cover of $\mathbb{P}_2$ branched along a sextic curve C_t, whose equation is given by the right hand side of (9). *The two-form ω_t transforms into* (10).

It is well-known (see e.g. [GH], p. 593) that a surface, which is a double cover of $\mathbb{P}_2$ branched along a smooth sextic curve is nothing but a $K3$ surface and the form (10) is a regular 2-form on it. In our case C_t has singularities. We will show that for $t\notin\{0,1,\infty,(\sqrt{2}\pm1)^4\}$ all of them are "simple", i.e. of *A-D-E* type (see [BPV], Ch. II. 8). These singularities are therefore harmless, the double cover of $\mathbb{P}_2$ branched along C_t has only *A-D-E* singularities. So the minimal resolution is a $K3$ surface and the form ω_t is still regular on it (see [BPV], Ch. III. 7).

In order to complete the proof of the main theorem we only have to investigate the singularities of the curve

$$C_t=\{(u,v,w)\in\mathbb{P}_2;\ 4tuv(w-u)^2(w-v)^2+w^2\big(u(w-v)t-v(w-u)\big)^2=0\}$$

and to determine the Picard number of the $K3$ surface

X_t = minimal resolution of singularities of the double cover of $\mathbb{P}_2$ branched along C_t.

(i) *Singularities of C_t* $(t\notin\{0,1,\infty,(\sqrt{2}\pm1)^4\})$. The points $P_1=(1,0,0)$, $P_2=(0,1,0)$, $P_3=(0,0,1)$ and $P_4=(1,1,1)$ are seen to be singular on C_t. Using for example the procedure of [BPV], Ch. II. 8, suitable local analytic coordinates can be found such that the equation of C_t near the points P_j has a particularly simple form. We omit these calculations, but state the result in the following table. See also Fig. 1.

singular point	Normal form	Type	Tangent lines
$P_1=(1,0,0)$	$y(x^2-y^3)$	D_5	$v-w=0$ (cuspidal), $v=0$
$P_2=(0,1,0)$	$y(x^2-y^3)$	D_5	$u-w=0$ (cuspidal), $u=0$
$P_3=(0,0,1)$	x^2-y^3	A_2 (cusp)	$ut+v=0$
$P_4=(1,1,1)$	x^2-y^4	A_3 (tacnode)	$(v-w)t-(u-w)=0$

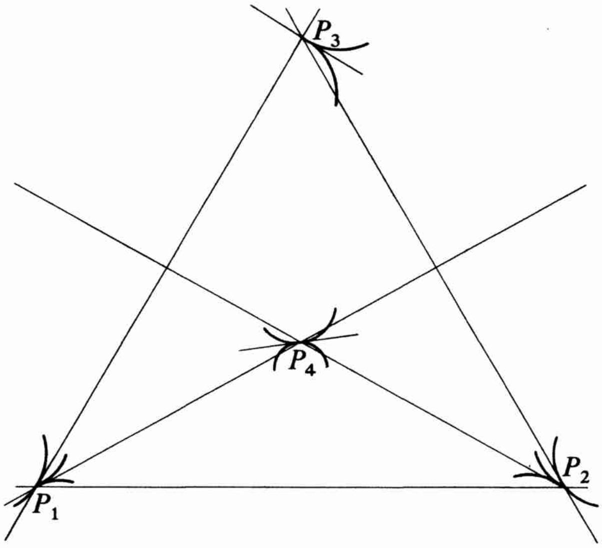

Figure 1

Notice that, except for P_3P_4, all lines connecting two singular points P_i, P_j intersect C_t only in these singular points. We claim that no other singularities are present. To establish this, we show first of all that

C_t is a (variable) elliptic curve for all $t \notin (0, 1, \infty, (\sqrt{2}\pm 1)^4)$.

We first go back to (x, y, t)-space. The curve $C_t \subset (z=0)$ is given by

$$4txy + (x+1)(y+1)(xt-y)^2 = 0.$$

By the substitution $y = u_1 x$ this curve is transformed into

$$4tu_1 + (x+1)(u_1x+1)(u_1-t)^2 = 0.$$

After the substitution $x_1 = u_1(t-u_1)x$ we obtain

$$x_1^2 + x_1\{(t-u_1)(u_1+1)\} + u_1(t+u_1)^2 = 0.$$

Finally, upon completing the square, we find with $X = u_1$ and $Y = x_1 + \frac{1}{2}(t-u_1)(u_1+1)$,

$$4Y^2 = (X-1)^2(X-t)^2 - 16tX^2.$$

This exhibits C_t as a double cover of $\mathbb{P}_1$ branched at the four points

$$\begin{cases} \frac{1}{2}(1+t+4\sqrt{t} \pm (1+\sqrt{t})\sqrt{1+t+6\sqrt{t}}), \\ \frac{1}{2}(1+t-4\sqrt{t} \pm (1-\sqrt{t})\sqrt{1+t-6\sqrt{t}}). \end{cases}$$

These four points are distinct, precisely when $t \notin \{0, 1, \infty, (\sqrt{2}\pm 1)^4\}$, so for these values C_t is elliptic, as claimed.

Secondly, we use the genus formula for plane curves C of degree d (cf. [BK], Ch. 9.2)

$$g(C)=\frac{1}{2}(d-1)(d-2)-\sum_{P}\frac{\mu_P(\mu_P-1)}{2}$$

where the summation is to be extended over all points P of C including the infinitely near ones at the singularities, and where μ_P is the multiplicity of P on C. For a discussion of infinitely near points and their multiplicities, see for example [BK], Ch. 8.4. The D_5-points P_1 and P_2 consist of one point with multiplicity 3, the cusp P_3 consists of one point with multiplicity 2 and the tacnode P_4 consists of two infinitely near points, each with multiplicity 2. Thus we obtain,

$$g(C_t)\leqq\frac{1}{2}\cdot 5\cdot 4-3-3-1-(1+1)=1.$$

Since we know that $g(C_t)=1$ we conclude that there are no other singularities on S besides $P_1,\ldots,P_4$.

(ii) *Determination of the Picard number $\rho(X_t)$.* We follow the process of [BPV], Ch. III. 7 to resolve the singularities of the double cover of $\mathbb{P}_2$ branched in C_t. We first apply the σ-process at all four singular points P_j simultaneously and we let E_j be the exceptional curve over P_j (see Fig. 2).

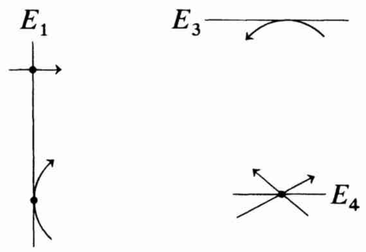

Figure 2

Let $\bar{C}_t$ be the proper transform of C_t. In Fig. 2 its branches near the singular points are the curves with arrows. We add those exceptional curves for which the multiplicity of the corresponding singularity on C_t is odd, i.e. we add E_1 and E_2 to get $C_t^{(1)}=\bar{C}_t+E_1+E_2$. Still, the double cover of the new surface branched along $C_t^{(1)}$ has singularities, since $C_t^{(1)}$ has singularities (the dots in Fig. 2). Repeating this procedure gives the less singular curve $C_t^{(2)}$ which is the proper transform of $C_t^{(1)}$ (see Fig. 3, the curve $C_t^{(2)}$ near the singularities consists of curves with arrows again).

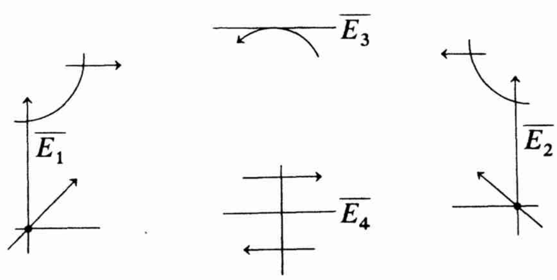

Figure 3

Still, this curve is not smooth, the dots in Fig. 3 are singular points. So we have to perform two more blow ups and we finally arrive at a surface Q on which the proper transform $C_t^{(3)}$ of $C_t^{(2)}$ is smooth (see Fig. 4 for the situation near P_1).

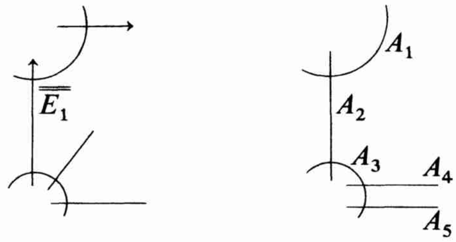

Figure 4

The double cover of Q branched along $C_t^{(3)}$ is the $K3$-surface X_t. In Fig. 5 we have drawn all curves on X_t lying above P_1, P_2, P_3, P_4 (see Fig. 4b for P_1 as well). The A-curves give the D_5 configuration over P_1, the B-curves give the D_5 over P_2, the curves C_1, C_2 give the A_2 over P_3 and the curves D_1, D_2, D_3 give the A_3 over P_4. We have also drawn the inverse images of the lines connecting two of the P_j's (except for P_3P_4). So M_{12} maps to the line connecting P_1 and P_2, the pairs $\{M_{ij}^{\pm}\}$ $((i,j)=(1,4), (2,4), (1,3), (23))$ map to the line connecting P_i and P_j.

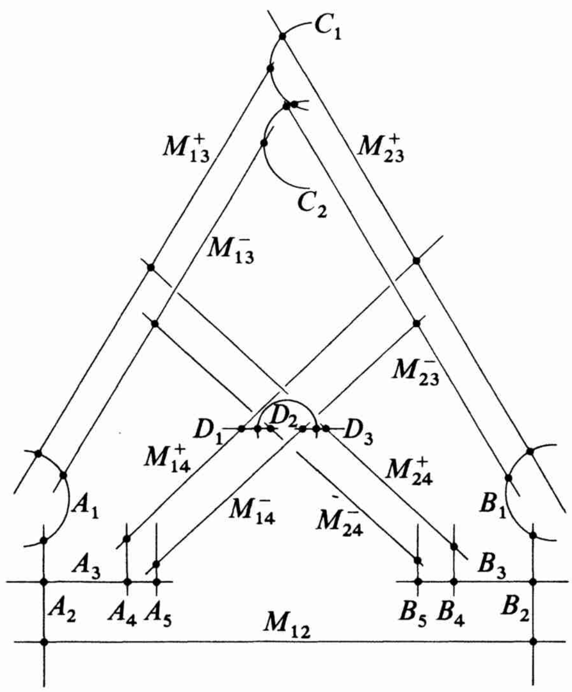

Figure 5

To determine the Picard number, we make use of the elliptic pencil on X_t coming from the lines through P_2. On X_t these lines have inverse images that are variable elliptic curves all meeting the curve B_2 in exactly one (variable) point. So B_2 gives a section for this elliptic fibration and we let 0 be the corresponding zero-element for the group of sections S. We let $m_{13}^{\pm}, m_{14}^{\pm} \in S$ be the elements corresponding to the curves $M_{13}^{\pm}, M_{14}^{\pm}$. The following observation is quite crucial for the determination of the Picard number,

For generic t, $m_{13}^{\pm}$ and $m_{14}^{\pm}$ have infinite order.

To prove this claim, we specialize to $u=1$, $w=-1$, $t=-1$ and then the elliptic curve in S_{-1} over the line $u+w=0$ can be given by

$$z^2 = 16v^3 - 23v^2 + 10v + 1.$$

This is a non-singular elliptic curve in the (z, v)-plane with a flex at infinity corresponding to 0. The section $m_{14}^{\pm}$ meets it in $(\pm 2, 1)$, the section $m_{13}^{\pm}$ in $(\pm 1, 0)$. The formula for doubling is easily derived:

$$v(2P) = -2v(P) + \frac{23}{16} + \frac{1}{16}\left(\frac{24v^2(P) - 23v(P) + 5}{z(P)}\right)^2.$$

Notice that $(\pm 2, 1) = 2 \cdot (\pm 1, 0)$. Thus it suffices to show that $T = (1, 0)$ has infinite order. Using the integral substitutions $x = 9 \cdot 16v - 3 \cdot 23$, $y = 27 \cdot 16z$ we bring the elliptic curve into normal form $y^2 = x^3 + Ax + B$, $A, B \in \mathbb{Z}$. Consider the point $4T$. Its (z, v) coordinates are $\left(\frac{289}{128}, \frac{69}{64}\right)$ and its (y, x) coordinates are $\left(\frac{27 \cdot 289}{8}, \frac{5}{4} \cdot 69\right)$. A consequence of the Nagell-Lutz theorem is, that a rational point on $y^2 = x^3 + Ax + B$ with non-integral coordinates has necessarily infinite order (see [M], Ch. XVI). Hence $4T$ has infinite order and the same is true for T.

We now proceed to show that $\rho(X_t) = 19$ for generic t. Since C_t is a variable elliptic curve, $\{X_t\}$ is a non-constant family of $K3$ surfaces, so not all X_t have maximal Picard number 20 (these form a discrete set in the moduli space, compare section 2). Hence $\rho(X_t) \leqq 19$ for generic t. Therefore it suffices to show that $\rho(X_t) \geqq 19$ for all

$$t \notin \{0, 1, \infty, (\sqrt{2} \pm 1)^4\}.$$

We apply [Sh 2], Corollary 1.5 which reads

$$\rho(X_t) = \text{rank of } S + 2 + \sum (m_\lambda - 1)$$

where the summation is over all reducible members F_λ of the elliptic pencil and where m_λ is the number of components of F_λ. From Fig. 5 some reducible members are easy to find,

$M_{12}, A_1, A_2, A_3, A_4, A_5$	(type I_1^*)	$m_\lambda = 6$,
$B_1, M_{23}^+, M_{23}^-, C_1, C_2$	(type I_5)	$m_\lambda = 5$,
$B_3, B_4, B_5, M_{24}^-, M_{24}^+, D_1, D_2, D_3$	(type I_8)	$m_\lambda = 8$.

The types refer to Kodaira's classification for singular fibres in an elliptic pencil, see e.g. [Sh 2], p. 36 or [Ko], p. 564. By the above claim we have rank $(S) \geqq 1$, hence

$$\rho(X_t) \geqq 1 + 2 + (5 + 4 + 7) = 19$$

as asserted.

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